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(posted)

f.) More Oscillations : Mechanics of Fields

→ recall the string : (i.e. continuum limit)

$$\mathcal{L} = \mathcal{L}(y, y_t, y_x) \quad \rightarrow \text{ Lagrangian density}$$

$$\mathcal{L} = \tfrac{1}{2}\mu\, y_t^2 - T\left[\underbrace{(1+y_x^2)^{1/2} - 1}_{\text{potential energy in string}}\right] \qquad \underline{\underline{(1D!)}}$$

where $S = \int_{t_1}^{t_2} dt \int_0^L dx\, \mathcal{L}$ $\qquad$ t, x both parameters

Then, for EOM: $\delta S = 0$ (as usual)

$$\delta S = 0 = \int_{t_1}^{t_2} dt \int_0^L dx \left(\frac{\partial \mathcal{L}}{\partial y}\delta y + \frac{\partial \mathcal{L}}{\partial y_t}\delta y_t + \frac{\partial \mathcal{L}}{\partial y_x}\delta y_x \right)$$

$$= \int_{t_1}^{t_2} dt \int_0^L dx \left(\frac{\partial \mathcal{L}}{\partial y}\delta y + \frac{\partial \mathcal{L}}{\partial y_t}\frac{d}{dt}\delta y + \frac{\partial \mathcal{L}}{\partial y_x}\frac{d}{dx}\delta y \right)$$

$$= \int_0^L dx\, \cancel{\frac{\partial \mathcal{L}}{\partial y_t}\delta y}\Big|_{t_1}^{t_2} \rightarrow 0 + \int_{t_1}^{t_2} dt\, \frac{\partial \mathcal{L}}{\partial y_x}\delta y\Big|_0^L$$

$$+ \int_{t_1}^{t_2}\int_0^L dx \left(\frac{\partial \mathcal{L}}{\partial y} - \frac{d}{dt}\left(\frac{\partial \mathcal{L}}{\partial y_t}\right) - \frac{d}{dx}\left(\frac{\partial \mathcal{L}}{\partial y_x}\right) \right)\delta y$$

fixed end pts in time!
↓
no config change.

thus, have Lagrange EOM:

$$\boxed{\begin{aligned} &\frac{d}{dt}\left(\frac{\partial \mathcal{L}}{\partial y_t}\right) = \frac{\partial \mathcal{L}}{\partial y} - \frac{d}{dx}\left(\frac{\partial \mathcal{L}}{\partial y_x}\right) \\ &\text{with B.C.:} \quad \left.\frac{\partial \mathcal{L}}{\partial y_x}\,\delta y\right|_0^L = 0 \end{aligned}}$$

(clear for fixed, free ends)

n.b.: → have

- spatial ibp endpt. $\displaystyle\int_{t_1}^{t_2} dt\, \left.\frac{\partial \mathcal{L}}{\partial y_x}\,\delta y\right|_0^L$

- $\delta y(t, x) = 0$, all x, only at t_2, t_1

<u>n.b.</u> free end

→ in 3D, have:

$$\frac{d}{dt}\left(\frac{\partial \mathcal{L}}{\partial y_t}\right) = \frac{\partial \mathcal{L}}{\partial y} - \frac{d}{dx_i}\left(\frac{\partial \mathcal{L}}{\partial x_i}\right)$$

$y \to \phi$
a scalar field

→ for 1D string:

$$\frac{d}{dt}\left(\mu y_t\right) = \frac{d}{dx}\left(\frac{T\, y_x}{(1+y_x^2)^{1/2}}\right)$$

small oscillations: $\mathcal{L} = \frac{1}{2}\mu y_t^2 - \frac{T}{2}(y_x)^2$

$\therefore$

$\mu y_{t,t} = T y_{xx}$ $\Rightarrow$ $\left\{ \begin{array}{l} \text{garden variety} \\ \text{wave eqn.} \end{array} \right.$

$\rightarrow$ Ex. $U(\phi) = \alpha \frac{\phi^2}{2} + \beta \phi^4$

$$\mathcal{L} = \frac{\phi_t^2}{2} - \frac{(\nabla\phi)^2}{2} - U(\phi)$$

Derive

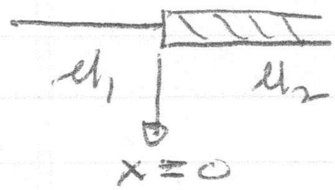
$\Rightarrow$ EOM ? $\Rightarrow$ K-G Eqn.
$\phi_{tt} - \phi_{xx} + \alpha\phi + \beta\phi^3 = 0$

$\rightsquigarrow$ acoustics
Aside: Standard Problems
Now, Lagrangian formulation allows unambiguous formulation of basic equations for matching;

$\Rightarrow$ consider $\underline{\underline{2}}$ prototypical examples

How to handle matching conditions?

i.e. $\left\{ \begin{array}{l} \text{matter of getting order of} \\ \text{derivatives correct} \end{array} \right.$

i.) 

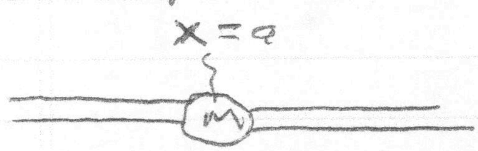
Junction $\Rightarrow$ un-equal mass

matching $\Rightarrow$ $y_-(0) = y_+(0)$

$$\int_{0_-}^{0_+} \left\{ \frac{d}{dt}\left(\frac{\partial \mathcal{L}}{\partial y_t}\right) - \frac{\partial \mathcal{L}}{\partial y} + \frac{d}{dx}\left(\frac{\partial \mathcal{L}}{\partial y_x}\right) \right\} = 0$$

i.e. integrate EOM

$$\left.\frac{\partial \mathcal{L}}{\partial y_x}\right|_{0_+} = \left.\frac{\partial \mathcal{L}}{\partial y_x}\right|_{0_-} \Rightarrow \text{slope match}$$

ii.) 

(continuity understood)

$\mu \to \mu + M\delta(x-a)$

$$\mathcal{L} = \frac{1}{2}\left(\mu + M\delta(x-a)\right) y_t^2 - \frac{T}{2} y_x^2$$

$$\left(\mu + M\delta(x-a)\right) y_{tt} = T y_{xx} \to \partial_x T \partial_x y$$

$$y = \tilde{y}(x) e^{-i\omega t}$$

4a.

$$T\hat{y}_{xx} = -\omega^2\left(\mu + M\delta(x-a)\right)\hat{y}$$

$$\int_{a_-}^{a_+}\left[T\hat{y}_{xx} + \omega^2\left(\mu + M\delta(x-a)\right)\hat{y}\right] = 0$$

$$T\frac{d\hat{y}}{dx}\Bigg|_{a_-}^{a_+} = -\omega^2 M\hat{y}(a)$$

$$T\left(\frac{d\hat{y}_+}{dx} - \frac{d\hat{y}_-}{dx}\right) = -\omega^2 M y(a)$$

$$T\hat{y}_x \Big|_{a_-}^{a_+} = -\omega^2 M\, y(a)$$

⇒ jump condition

⇒ Mass induces jump

N.B.: Use of Lagrangian ab-initio renders all questions re: order of derivatives moot.

→ <u>Hamiltonian</u> <u>Formulation</u>

Define <u>canonical</u> momentum:

$$\pi = \partial \mathcal{L}/\partial \dot{y} \rightarrow \mu \dot{y}$$

$$= \mu \dot{y}$$

→ momentum (in y) of string element

Can define Hamiltonian density:

$$\mathcal{H} = \pi \dot{y} - \mathcal{L}$$

$$H = \int_0^L dx\, \mathcal{H}$$

for $\frac{\partial \mathcal{L}}{\partial t} = 0$, $\mathcal{H} = \mathcal{E}$

↳ Hamiltonian.

For string:

$$\mathcal{H} = \frac{\pi^2}{\mu} - \mathcal{L} = \underbrace{\frac{\pi^2}{2\mu}}_{\text{Kin E.}} + \underbrace{\frac{T}{2} y_x^2}_{\text{Pot E}}$$

e.g.

→ Lagrangian formulation of acoustics

- seek formulate acoustic wave from Lagrangian perspective (why? – systematics, i.e. P.T. for NL.)

- useful to define generalized coordinate

$$\underline{\eta} = \underline{\xi} = (\xi_x, \xi_y, \xi_z)$$

↔ displacement of gas element (Lagrangian

obviously: $\partial_t \underline{\eta} = \underline{V}$ → gas element velocity ✓

and continuity (i.e. no gas lost in oscillation – mass conservation) ⇒

$$\partial_t \rho + \underline{\nabla} \cdot (\rho \underline{V}) = 0$$

(akin $\partial_t \rho + \underline{\nabla} \cdot \underline{J} = 0$, $\underline{J} \to \rho \underline{V}$)

then $\partial_t \tilde{\rho} = -\underline{\nabla} \cdot (\rho_0 \partial_t \tilde{\underline{\eta}})$

$$\tilde{\rho} = -\underline{\nabla} \cdot (\rho_0 \tilde{\underline{\eta}}) = -\rho_0 \underline{\nabla} \cdot \tilde{\underline{\eta}} \quad \text{(homog.)}$$

(displacement ↔ compression relation).

- obviously need $\mathcal{L}$, where $\mathcal{L} = \mathcal{T} - \mathcal{U}$

$\mathcal{T}$ = kinetic energy density

$$\mathcal{T} = \frac{\rho_0 \dot{\eta}^2}{2}$$

Issue is $\mathcal{U}$? (could guess from energetics)

Now, extension of "string" calculation to scalar field in 3D $\Rightarrow$ LEOM

$$\frac{d}{dt}\left(\frac{\partial \mathcal{L}}{\partial \dot{\eta}}\right) + \sum_{k=1}^{3} \frac{d}{dx_k}\left(\frac{\partial \mathcal{L}}{\partial(\partial \eta / \partial x_k)}\right) - \frac{\partial \mathcal{L}}{\partial \eta} = 0$$

will seek $\dfrac{\partial \mathcal{L}}{\partial(\partial \eta / \partial x_k)}$

Now, for $\mathcal{U}$, consider gas physics:

$V_0 = M / \rho_0$ → volume

$$\mathcal{U} V_0 = -\int_{V_0}^{V_0 + \Delta V} P \, dV$$

potential energy from compression/expansion

work performed on system (incr. in P.E.)

$$\text{Now} \int_{V_0}^{V_0 + \Delta V} P \, dV = P_0 \Delta V + \left.\frac{\partial P}{\partial V}\right)_0 \frac{(\Delta V)^2}{2} + \text{h.o.t.}$$

Now $P \rho^{-\gamma} = \text{const}$, $P V^{\gamma} = \text{const}$ } eqn of state

N.B. Sound wave too fast for isothermality
$\Rightarrow$ conduction too slow.

$$\left(\frac{\partial p}{\partial V}\right)_0 = -\frac{\gamma p}{V}$$

Now $\delta\rho = -\rho_0(\nabla \cdot \vec{\eta})$

$$\frac{\Delta V}{V_0} = -\frac{\delta\rho}{\rho}$$

$$\left(\rho = 1/V \rightarrow \delta\rho = -\frac{\delta V}{V^2}, \quad \delta\rho/\rho = -\delta V/V\right)$$

so

$$\mathcal{U} V_0 = -\left[p_0 \Delta V + \left(\frac{\partial p}{\partial V}\right)_0 \frac{(\Delta V)^2}{2} \right]$$

$$= \left(\frac{\gamma p_0}{V_0}\right) \frac{1}{2} V_0^2 \left(\frac{\delta\rho}{\rho_0}\right)^2$$

$$+ p_0 V_0 \frac{\delta\rho}{\rho}$$

$$\mathcal{U} = \frac{\gamma p_0}{2} \left((\nabla \cdot \vec{\eta})^2\right) - p_0 \nabla \cdot \vec{\eta}$$

Finally,

$$\mathcal{U} = \frac{\gamma p_0}{2} (\nabla \cdot \vec{\eta})^2 - p_0 (\nabla \cdot \vec{\eta})$$

$$\mathcal{L} = \frac{1}{2} \rho_0 \dot{\vec{\eta}}^2 - \frac{\gamma p_0}{2} \left[(\nabla \cdot \vec{\eta})^2 - p_0 (\nabla \cdot \vec{\eta}) \right]$$

$$\frac{\partial \mathcal{L}}{\partial \eta_i} = 0$$

1D - simplicity:

$$\mathcal{L} = \frac{1}{2} \rho_0 \dot{\eta}^2 - \frac{\gamma P_0}{2} \left(\left(\frac{\partial \eta}{\partial x}\right)^2 + P_0 \left(\frac{\partial \eta}{\partial x}\right) \right)$$

$$\frac{d}{dt}\left(\frac{\partial \mathcal{L}}{\partial \dot{\eta}}\right) + \frac{d}{dx}\left(\frac{\partial \mathcal{L}}{\partial (\partial \eta / dx)}\right) = 0$$

$$\frac{d}{dt}\left(\rho_0 \dot{\eta}\right) + \frac{d}{dx}\left(-\frac{\gamma P_0}{\cancel{2}} \frac{\partial \eta}{\partial x} + \underbrace{\cancel{P_0}}_{\text{const.}}\right) = 0$$

$\Rightarrow$

$$\rho_0 \ddot{\eta} - \gamma P_0 \frac{\partial^2 \eta}{\partial x^2} = 0 .$$

Wave eqn

in 3D,

$$\rho_0 \frac{\partial^2 \vec{\eta}}{\partial t^2} - \gamma P_0 \nabla(\nabla \cdot \vec{\eta}) = 0 .$$

- derive from fluid eqns.

So, as before, Hamilton's Eqns for continuua follow from Principle of Least Action:

$$S = \int_{t_1}^{t_2} dt \int_0^L dx \left(\pi \dot{y} - \mathcal{H} \right)$$

$$\mathcal{L} = \mathcal{L}(\dot{y}, y_x, y)$$
$$\mathcal{H} = \mathcal{H}(\pi, y_x, y)$$
$$\mathcal{L} = \pi \dot{y} - \mathcal{H}$$

so

$$\delta S = \int_{t_1}^{t_2} dt \int_0^L dx \left(\pi \, \delta \dot{y} + \dot{y} \, \delta \pi - \left(\frac{\partial \mathcal{H}}{\partial \pi} \delta \pi + \frac{\partial \mathcal{H}}{\partial y_x} \delta y_x + \frac{\partial \mathcal{H}}{\partial y} \delta y \right) \right)$$

ignoring surface terms:

$$= \int_{t_1}^{t_2} dt \int_0^L dx \left\{ \dot{y} \, \delta \pi - \dot{\pi} \, \delta y - \frac{\partial \mathcal{H}}{\partial y} - \frac{\partial \mathcal{H}}{\partial \pi} \delta \pi - \frac{\partial \mathcal{H}}{\partial y_x} \delta y_x \right\}$$

regrouping

$$\delta S = \int_{t_1}^{t_2} dt \int_0^L dx \left\{ \delta y \left(\frac{d\pi}{dt} + \frac{\partial \mathcal{H}}{\partial y} - \frac{d}{dx}\left(\frac{\partial \mathcal{H}}{\partial y_x}\right)\right) + \delta\pi \left(\dot{y} - \frac{\partial \mathcal{H}}{\partial \pi} \right) \right\}$$

so $\delta S = 0 \Rightarrow$

$$\boxed{\begin{cases} \dot{y} = \dfrac{\partial \mathcal{H}}{\partial \pi} \\ \dot{\pi} = -\dfrac{\partial \mathcal{H}}{\partial y} + \dfrac{d}{dx}\left(\dfrac{\partial \mathcal{H}}{\partial y_x}\right) \end{cases}}$$

Hamilton's Eqns $\Rightarrow$ motion of elements parametrized by x, t.

Now, $\partial \mathcal{L}/\partial t = \partial \mathcal{H}/\partial t = 0$ here

$$\mathcal{H} = \pi \dot{y} - \mathcal{L}$$

$$\frac{d\mathcal{H}}{dt} = \pi \ddot{y} + \dot{y}\dot{\pi} - \frac{d\mathcal{L}}{dt}$$

$$= \pi \ddot{y} + \dot{y}\dot{\pi} - \left(\cancel{\frac{\partial \mathcal{L}}{\partial t}} + \frac{\partial \mathcal{L}}{\partial y}\dot{y} + \frac{\partial \mathcal{L}}{\partial \dot{y}}\ddot{y} + (\partial \mathcal{L}/\partial y_x)\,\dot{y}_x \right)$$

but, $\pi = \partial \mathcal{L} / \partial \dot{y}$

so $\pi \ddot{y}$ cancels $-(\partial \mathcal{L} / \partial \dot{y}) \ddot{y}$

$\Rightarrow$

$$\frac{dH}{dt} = \dot{\pi} \dot{y} - \left(\frac{\partial \mathcal{L}}{\partial y} \dot{y} + \frac{\partial \mathcal{L}}{\partial y_x} \dot{y}_x \right)$$

and from LEOM:

$$\frac{d}{dt} \left(\frac{\partial \mathcal{L}}{\partial \dot{y}} \right) + \frac{d}{dx} \left(\frac{\partial \mathcal{L}}{\partial y_x} \right) = \frac{\partial \mathcal{L}}{\partial y}$$

so

$\partial \mathcal{L} / \partial \dot{y} = \pi$

$$\frac{dH}{dt} = \cancel{\dot{\pi} \dot{y}} - \dot{y} \left(\cancel{\frac{d}{dt} \left(\frac{\partial \mathcal{L}}{\partial \dot{y}} \right)} + \frac{d}{dx} \left(\frac{\partial \mathcal{L}}{\partial y_x} \right) \right) - \frac{\partial \mathcal{L}}{\partial y_x} \dot{y}_x$$

$$= -\dot{y} \frac{d}{dx} \left(\frac{\partial \mathcal{L}}{\partial y_x} \right) - \frac{\partial \mathcal{L}}{\partial y_x} \frac{d}{dx} \dot{y}$$

$\Rightarrow$ so finally regrouping:

$$\boxed{\frac{d\mathcal{H}}{dt} + \frac{d}{dx}\left(\dot{y}\frac{\partial \mathcal{L}}{\partial y_x}\right) = 0}$$

What does it mean?

→ Here $\mathcal{H} = \mathcal{E}$

so above ⟺

$$\frac{d}{dt}\mathcal{E} + \frac{d}{dx}\mathcal{S}_x = 0$$

$\mathcal{E}$ → excitation energy density

$\mathcal{S}$ → excitation energy density flux!

check: $\mathcal{S}_x = \dot{y}\dfrac{\partial \mathcal{L}}{\partial y_x}$

$= -\dot{y}\, y_x T$

$\omega^2 = c^2 k^2$

$c^2 = T/\mu$

$y = A \sin(kx - \omega t)$

$= +A^2 k\omega T \cos^2(kx - \omega t)$

$= k\omega T A^2 \cos^2(kx - \omega t)$

$\omega = ck$

$$\bar{S}_x = \frac{\omega^2}{c} T \frac{A^2}{2} = \frac{\omega^2}{c^2} T c \frac{A^2}{2}$$

phase/grp velocity (dispersionless)

$$= c\,\mathcal{E}$$

$\hookrightarrow$ energy density

$\Rightarrow$ wave energy density flux

$\frac{g_{vs}}{\phi}$ $2 \cdot \frac{1}{2}\left(\frac{1}{2}\right)$

so

$$\left.\frac{\partial \mathcal{E}}{\partial t} + \frac{\partial}{\partial x} S_x = 0\right]$$

$$S = c\,\mathcal{E} \rightarrow v_{gr}\,\mathcal{E}$$

Result is a "Poynting Thm." for string

In higher dims:

$$\partial_t \mathcal{E} + \underline{\nabla} \cdot \underline{S} = 0$$

and if external work done on system
or dissipation (i.e. internal friction)

$$\partial_t \mathcal{E} + \underline{\nabla} \cdot \underline{S} + P_{diss} = 0$$

Note:

→ Poynting thm. relates (local) wave energy density with wave energy density flux, i.e.

$$\frac{d\mathcal{H}}{dt} + \partial_x S_x = 0$$

→ Poynting thm. relates rate of energy change to wave energy density flux thru interval

i.e.

$$\frac{d}{dt} E = \frac{d}{dt}\int_{x_1}^{x_2} \mathcal{H}\, dx = -\int_{x_1}^{x_2} \frac{\partial}{\partial x} S_x$$

$$= -S_x \Big|_{x_1}^{x_2}$$

→ Poynting thm. formed by expressing $\frac{d}{dt}\mathcal{E}$ as $\underline{\nabla}\cdot\underline{S}$, etc.

recall in E and M:

$$\underline{\nabla}\times\underline{B} = \frac{4\pi}{c}\underline{J} + \frac{1}{c}\frac{\partial \underline{E}}{\partial t}$$

$$\underline{\nabla}\times\underline{E} = -\frac{1}{c}\,\partial\underline{B}/\partial t$$

 [redacted]

but $\mathcal{E} = E^2/8\pi + B^2/8\pi$

then $\left(\frac{\partial \underline{E}}{\partial t} = c\,\underline{\nabla}\times\underline{B} - 4\pi\,\underline{J}\right) \cdot \underline{E}/4\pi$

$\left(\frac{\partial \underline{B}}{\partial t} = -c\,\underline{\nabla}\times\underline{E}\right) \cdot \left(\underline{B}/4\pi\right)$

local power dissipated.
Analogue for string $\dot{I}$

$\Rightarrow$

$$\frac{\partial}{\partial t}\left(\frac{E^2+B^2}{8\pi}\right) = -\underline{E}\cdot\underline{J} - \underline{\nabla}\cdot\underbrace{\left(\frac{c}{4\pi}\,\underline{E}\times\underline{B}\right)}_{\underline{S}}$$

i.e. form Poynting thm. by considering time rate of change of energy density.

→ Important to distinguish:

(particle)

$\underline{\Pi} = \rho\dot{y}\hat{y} \equiv$ canonical momentum

→ momentum of string element $\rho\dot{y}(x,t)$, in $\hat{y}$ direction

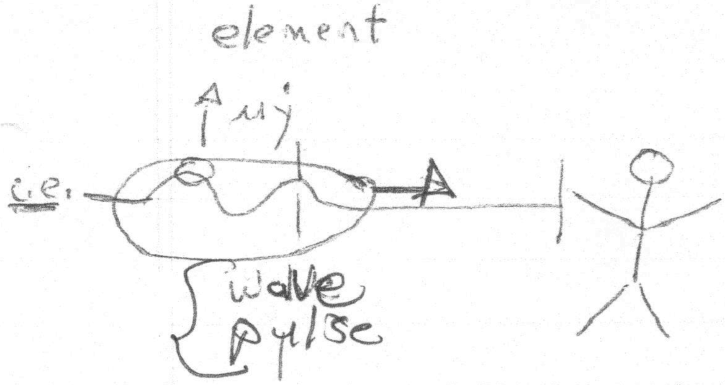
(quasi-particle)

$$\underline{S} = -T\frac{\partial y}{\partial t}\frac{\partial y}{\partial x}\hat{x} = \frac{\partial \mathcal{L}}{\partial y_x}\frac{\partial y}{\partial t}\hat{x}$$

$\equiv$ wave energy density flux

→ related to momentum of wave/fluctuation, in $\hat{x}$ direction

13. [redacted]



→ "feels" wave momentum (kick in $\vec{x}$)

flux thru (x) at time t $= \underline{S_x(x,t)}$

calculating for wave on string:

if $y = A\cos(k(x - v_{ph}t))$

$$v_{ph} = (T/\mu)^{1/2}$$

$$\frac{\partial y}{\partial t} = +Akv_{ph}\sin(k(x - v_{ph}t))$$

$$\frac{\partial y}{\partial x} = -Ak\sin(k(x - v_{ph}t))$$

$$S_x = +TA^2k^2v_{ph}\sin^2(k(x - v_{ph}t))$$

$$\therefore \overline{S_x} = \frac{Tk^2}{2}v_{ph}A^2$$

but: $\omega^2 = v_{ph}^2 k^2$

$$\boxed{\overline{S_x} = \frac{\mu\omega^2}{2}v_{ph}A^2}$$

$$\overline{S_x} = v_{gr}\mathcal{E}$$

as $v_{gr} = v_{ph}$

$$\mathcal{E} = 2 * \overline{KE}$$

$$= 2 * \frac{1}{4}\mu\omega^2A^2$$

→ Wave Momentum Density

- have developed notions of wave energy and Poynting Theorem, i.e.

- natural to investigate wave momentum density

Now, recall in E+M,

$$\underline{P}_{EM} = \frac{1}{c^2}\underline{S} = \frac{1}{4\pi c}\underline{E}\times\underline{B} = \frac{1}{c^2}\left(\begin{array}{c}\text{Wave Energy}\\ \text{Density Flux}\end{array}\right)$$

$\underline{S}$ ↳ Poynting vector

$\underline{P}_{EM}$ ↓ momentum of electromagnetic wave

Thus, natural motivation to investigate relation for string, i.e.

$$p = \dot{y}\frac{\partial \mathcal{L}}{\partial y_x}$$

so

$$\dot{p} = \ddot{y}\frac{\partial \mathcal{L}}{\partial y_x} + \dot{y}\frac{d}{dt}\left(\frac{\partial \mathcal{L}}{\partial y_x}\right)$$

for string;

$$\ddot{y} = \frac{T}{\mu}y_{xx} = V_{ph}^2 y_{xx} \quad ; \quad \frac{\partial \mathcal{L}}{\partial y_x} = -T y_x$$
$$= -T y_x$$

$$\Rightarrow \quad \dot{S}_x = \left\{ -\frac{T}{\mu} y_{xx} T \cdot y_x - \mu \dot{y} \frac{T}{\mu} \dot{y}_x \right\}$$

$$= -\frac{T}{\mu} \frac{\partial}{\partial x} \left\{ \frac{T y_x^2}{2} + \frac{\mu \dot{y}^2}{2} \right\}$$

$$= -c^2 \frac{\partial}{\partial x} \mathcal{E}$$

then, have:

$$\frac{\partial}{\partial t} S_x + c^2 \frac{\partial}{\partial x} \mathcal{E} = 0$$

so, if ala E+M:

$$c^2 = T/\mu$$

$$P_w = S/c^2$$

wave momentum density

$$\Rightarrow \boxed{\frac{\partial}{\partial t} P_w + \frac{\partial}{\partial x} \mathcal{E} = 0}$$

in 1D

$$\boxed{\frac{\partial}{\partial t} P_W + \nabla_x \mathcal{H} = 0}$$

1D, only

here $\nabla_x \mathcal{H} = \nabla_x \mathcal{E}$ is force density

pushes in direction of propagation

$$\underline{P_W} = \int_{x_1}^{x_2} dx \, P_W$$

momentum in wave packet in pulse of string $[x_1, x_2]$

so:

$$\frac{\partial}{\partial t} P_W = -\mathcal{H} \Big|_{x_1}^{x_2}$$

difference / jump in energy density across the chunk of string

$\Rightarrow$ net change in WMD

Note:

a.) Semi-classical analogy

$$\mathcal{E} = \omega\, \mathcal{E}/\omega = N\omega$$

↳ wave action density (see next lecture)

$$P_W = \frac{S}{c^2} = \frac{k}{\omega}\,\frac{\cancel{c}}{\cancel{c}}\,\mathcal{E}$$

$$= \frac{k}{\cancel{\omega}}\, N\cancel{\omega} = kN$$

Wave energy density → $N\omega$

Wave momentum density → Nk

N → # waves / wave population density.

ii) $m\dot{y} = \pi$ → canonical momentum $\hat{y}$ direction

Now <u>symmetry</u> connection:

a) if string + disturbance translated in $\hat{x}$ and result invariant:

$\Rightarrow \exists$ conserved <u>momentum</u>

but

(b) if disturbance / pulse translated in x with string fixed, and result invariant:

$\Rightarrow$ $\exists$ conserved Pseudomomentum.

$$\boxed{\text{Evidently Pseudomomentum} \leftrightarrow \text{Wave Momentum Density!}}$$

(iii) Note can write:

$$\frac{\partial}{\partial t} \mathcal{H} + \frac{\partial}{\partial x} S = 0$$

$$\frac{\partial}{\partial t} P_w + \frac{\partial}{\partial x} \mathcal{H} = 0$$

$$\Rightarrow \left(\frac{1}{v_{ph}} \frac{\partial}{\partial t}, \frac{\partial}{\partial x} \right) \begin{bmatrix} \mathcal{H}, & S/v_{ph} \\ S/v_{ph} & \mathcal{E} \end{bmatrix} = 0$$

i.e. can think as:

$$\partial_\mu T^{\mu\nu} = 0$$

$T^{\mu\nu}$ = energy-momentum tensor of string

$$T^{\mu\nu} = \begin{bmatrix} \mathcal{H}, & S/v_{ph} \\ S/v_{ph}, & \mathcal{H} \end{bmatrix}$$

$$\partial_\mu = \left(1/v_{ph}\,\partial_t,\ \partial x\right)$$

For EM:

$$T^{ik} = \begin{pmatrix} \overset{(E^2+H^2)/8\pi}{W}, & S_x/c, & S_y/c, & S_z/c \\ S_x/c & & & \\ S_y/c & & \underline{\underline{\sigma}} & \\ S_z/c & & & \end{pmatrix}$$

$$\sigma_{\alpha,\beta} = 1/4\pi \left\{ -E_\alpha E_\beta - H_\alpha H_\beta + \frac{\delta_{\alpha,\beta}}{2}\left(E^2 + H^2\right) \right\}$$

π

Maxwell stress tensor.

⇒ Application : Sound

$$\partial_t \rho + \underline{\nabla} \cdot (\rho \underline{v}) = 0$$

$$\frac{\partial}{\partial t} \underline{v} + \underline{v} \cdot \underline{\nabla} \underline{v} = -\frac{\underline{\nabla} p}{\rho}$$

linearizing ⇒

$$\frac{\partial \tilde{v}}{\partial t} = -\frac{c_s^2}{\rho_0} \underline{\nabla} \tilde{\rho}$$

$$p = p(\rho)$$
$$dp/d\rho = c_s^2$$

$$\frac{\partial}{\partial t} \tilde{\rho} = -\rho_0 \underline{\nabla} \cdot \underline{v}$$

then:

$$\frac{\partial^2}{\partial t^2} \tilde{\rho} = \rho_0 \underline{\nabla} \cdot \left\{ \frac{c_s^2}{\rho_0} \underline{\nabla} \tilde{\rho} \right\} = c_s^2 \nabla^2 \tilde{\rho}$$

$$\partial_t^2 \tilde{\rho} = c_s^2 \nabla^2 \tilde{\rho}$$

⇒ wave eqn.

then: $$\frac{\partial^2 \hat\rho}{\partial t^2} = c_s^2 \nabla^2 \hat\rho = \rho_0 \nabla \cdot \left\{ \frac{c_s^2}{\rho_0} \nabla \rho \right\}$$

For energy-momentum relations:

$$(1) \cdot \hat{V}\rho_0 + (2) \cdot \frac{\hat\rho c_s^2}{\rho_0} \Rightarrow \hat{V} \cdot \nabla \hat\rho$$

$$\frac{\partial}{\partial t}\left(\frac{\rho_0 \hat{V}^2}{2}\right) + c_s^2 \lambda = 0$$

$$\frac{\partial}{\partial t}\left(\frac{\hat\rho^2 c_s^2}{2\rho_0}\right) + c_s^2 \hat\rho \, \nabla \cdot \hat{V} = 0$$

$$\therefore \frac{\partial}{\partial t}\left(\frac{\rho_0 \hat{V}^2}{2} + \frac{\hat\rho^2 c_s^2}{2\rho_0}\right) + \nabla \cdot \Big[\underbrace{c_s^2 \rho V}_{S} \Big] = 0$$

$$H = \mathcal{E} = \underbrace{\frac{\rho_0 \hat{V}^2}{2}}_{T} + \underbrace{\frac{\hat\rho^2 c_s^2}{2\rho_0}}_{V} \qquad S \rightarrow \text{acoustic wave energy density flux}$$

$T \rightarrow$ fluid motion

$V \rightarrow$ (compression)

$$U = \left(\frac{\tilde\rho}{\rho_0}\right)^2 \frac{\rho_0 c_s^2}{2} = \frac{\gamma P_0}{2} \left(\tilde\rho / \rho_0\right)^2$$

Similarly,

$$P_w = \frac{1}{c_s^2} S$$

$$\frac{\partial P_w}{\partial t} = \frac{\partial}{\partial t}(\rho V) = \frac{\partial \hat\rho}{\partial t} \hat{V} + \hat\rho \frac{\partial \hat{V}}{\partial t}$$

but $\tilde\rho/\rho_0 = -\nabla \cdot \eta$ → recovers previous

$$\Rightarrow \quad \frac{\partial \tilde{\rho}}{\partial t} = -\rho_0 \underline{\nabla} \cdot \underline{V} \qquad (1)$$

$$\frac{\partial \underline{\tilde{V}}}{\partial t} = -\frac{C_s^2}{\rho_0} \underline{\nabla} \tilde{\rho} \qquad (2)$$

$$\underline{V} \, (1) + \tilde{\rho} \, (2) \Rightarrow$$

$$\frac{\partial}{\partial t}(\tilde{\rho} \underline{\tilde{V}}) = -\rho_0 \underline{V} (\underline{\nabla} \cdot \underline{V}) - \frac{C_s^2}{2\rho_0} \nabla (\tilde{\rho}^2)$$

$$= -\underline{\nabla} \left(\frac{\rho \tilde{V}^2}{2} + \frac{C_s^2}{\rho_0} \frac{\tilde{\rho}^2}{2} \right)$$

[Momentum Density relation for longitudinal linear waves.

$$\frac{\partial}{\partial t} \frac{\underline{S}}{C_s^2} = -\underline{\nabla} \, \Sigma$$

ignores all but linear wave energy

$$= \frac{\partial}{\partial t} \underline{P_w}.$$